

FOLIATIONS WITH ALL NON-CLOSED LEAVES ON NON-COMPACT SURFACES

SERGIY MAKSYMENKO AND EUGENE POLULYAKH

ABSTRACT. Let X be a connected non-compact 2-dimensional manifold possibly with boundary and Δ be a foliation on X such that each leaf $\omega \in \Delta$ is homeomorphic to \mathbb{R} and has a trivially foliated neighborhood. Such foliations on the plane were studied by W. Kaplan who also gave their topological classification. He proved that the plane splits into a family of open strips foliated by parallel lines and glued along some boundary intervals. However W. Kaplan's construction depends on a choice of those intervals, and a foliation is described in a non-unique way. We propose a canonical cutting by open strips which gives a uniqueness of classifying invariant. We also describe topological types of closures of those strips under additional assumptions on Δ .

1. INTRODUCTION

Let X be a 2-dimensional manifold possibly non-connected and having a boundary, and Δ be a one-dimensional foliation on X . We will say that Δ belongs to class \mathcal{F} if it satisfies the following three conditions.

- (1) Each leaf ω of Δ is a closed subset of X .
- (2) Every connected component ω of ∂X is a leaf of Δ .
- (3) Let $\omega \in \Delta$ be a leaf, and $J = [0, 1)$ if $\omega \subset \partial X$, and $J = (-1, 1)$ otherwise. Then there exists an open neighborhood U of ω and a homeomorphism $\phi : \mathbb{R} \times J \rightarrow U$ such that $\phi(\mathbb{R} \times 0) = \omega$ and $\phi(\mathbb{R} \times t)$ is a leaf of Δ for all $t \in J$, see Figure 1.1.

Roughly speaking, a 1-dimensional foliation Δ is a partition of X which looks like a partition of \mathbb{R}^2 into parallel lines *near each point* $x \in X$. Then Δ belongs to class \mathcal{F} whenever it looks like partition of \mathbb{R}^2 into parallel lines *near each leaf* $\omega \in \Delta$. In particular, each leaf of Δ is homeomorphic to \mathbb{R} .

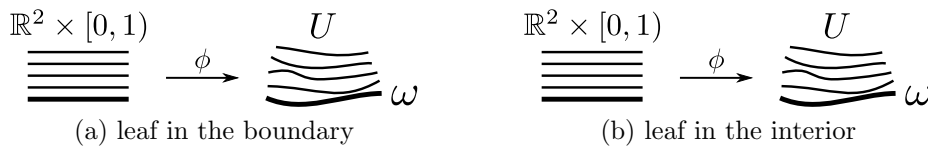


FIGURE 1.1.

Definition 1.1. Let X_i be a surface with a foliation Δ_i , $i = 1, 2$. Then a homeomorphism $h : X_1 \rightarrow X_2$ will be called *foliated* if it maps leaves of Δ_1 onto leaves of Δ_2 . In this case we will also write $h : (X_1, \Delta_1) \rightarrow (X_2, \Delta_2)$.

The aim of the present paper is to describe a topological structure of foliations belonging to class \mathcal{F} up to foliated homeomorphisms, see Theorem 1.8 below. Such foliations on the plane were studied by W. Kaplan [13] and they appear as foliations by level sets

2010 *Mathematics Subject Classification.* 57R30, 55R10.

Key words and phrases. Foliation, non-compact surface, fiber bundles.

of pseudoharmonic functions on \mathbb{R}^2 , see W. Kaplan [13, Theorem 42], W. Boothby [6], [7], M. Morse and J. Jenkins [12], M. Morse [16]. We will improve Kaplan's construction and extend it to foliations on arbitrary surfaces.

Topological structure of singular foliations on surfaces, in particular, foliations by orbits of flows, were studied by A. Andronov and L. Pontryagin [1], M. Peixoto [18], [19], S. Aranson and V. Grines [2, 3], I. Bronstein and I. Nikolayev [8], S. Aranson, E. Zhuzhoma, and V. Medvedev [5], L. Plachta [22, 20, 21], A. Oshemkov and V. Sharko [17], S. Aranson, V. Grines and V. Kaimanovich [4], M. Farber [11], N. Budnytska and O. Prishlyak [9], N. Budnyts'ka and T. Rybalkina [10] and many others. Results of the paper could also be applied to singular foliations without non-closed leaves on surfaces by removing singularities. This will be done in subsequent papers by the authors.

Special leaves. Suppose Δ is a foliation of class \mathcal{F} on a surface X . Let $Y = X/\Delta$ be the space of leaves, and $p : X \rightarrow Y$ be the corresponding quotient map. Endow Y with the *quotient topology*, so a subset $V \subset Y$ is open if and only if its inverse $p^{-1}(V)$ is open in X . For a subset $U \subset X$ its *saturation*, $S(U)$, with respect to Δ is the union of all leaves of Δ intersecting U . Equivalently, $S(U) = p^{-1}(p(U))$.

Since each leaf of Δ is a closed subset of X , it follows that Y is a T_1 -space. However, in general, Y is not a Hausdorff space.

Lemma 1.2. *If $\Delta \in \mathcal{F}$ then the projection map $p : X \rightarrow Y$ is open.*

Proof. We have to prove that for each open $V \subset X$ its saturation $S(V)$ is open as well. Thus for each $x \in S(V)$ we should find an open saturated subset W such that $x \in W = S(W) \subset S(V)$. Let ω be the leaf containing x . Put $J = [0, 1)$ whenever $\omega \subset \partial X$ and $J = (-1, 1)$ otherwise. Then by definition of class \mathcal{F} there exists a foliated homeomorphism $\phi : \mathbb{R} \times J \rightarrow U$ such that $\phi^{-1}(x) = (t, 0) \in \mathbb{R} \times 0$ for some $t \in \mathbb{R}$. Then $\phi^{-1}(V \cap U)$ is an open neighborhood of $(t, 0)$, whence there exists $\varepsilon > 0$ such that if we denote $K = J \cap (-\varepsilon, \varepsilon)$, then $t \times K \subset \phi^{-1}(V \cap U)$. But K is open in J , whence $\mathbb{R} \times K$ is open in $\mathbb{R} \times J$. Therefore $\phi(\mathbb{R} \times K)$ is saturated and open in U which in turn is open in X . Hence $\phi(\mathbb{R} \times K)$ is open in X and $x \in \phi(\mathbb{R} \times K) \subset S(V)$. Therefore $S(V)$ is open in X . \square

Definition 1.3. Let ω be a leaf of Δ and $y = p(\omega) \in Y$. We will say that ω is a *special* leaf and y is a *special* point of Y whenever Y is not Hausdorff at y , that is $y \neq \bigcap_{V \in \mathcal{V}} \overline{V}$, where V runs over all open neighborhoods of y .

Example 1.4. Consider the foliation on \mathbb{R}^2 shown in Figure 1.2(a). It splits by bold leaves α, β, γ , and δ into five “strips” A, B, C, D, E foliated by “parallel” lines, see Figure 1.2(b). Moreover, the space of leaves Y has the structure as in Figure 1.2(c), where bold lines correspond to strips, and thin lines just indicate that α belongs to the closure of A and B , β belongs to the closures B and C and so on. In particular, Y loses Hausdorff property at α, β, γ , and δ . More precisely, the subspace $Y \setminus \{\alpha, \beta, \gamma, \delta\}$ is Hausdorff, however each neighborhood of α intersect each neighborhood of β , and the same holds for pairs $\{\beta, \gamma\}$ and $\{\gamma, \delta\}$. Therefore the leaves α, β, γ and δ are special.

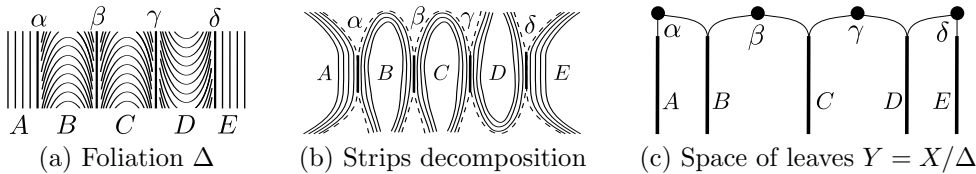


FIGURE 1.2.

Definition 1.5. A subset $S \subset \mathbb{R}^2$ will be called a *model strip* if there exist $a < b$ such that

- (1) $\mathbb{R} \times (a, b) \subset S \subset \mathbb{R} \times [a, b]$;
- (2) the intersection $S \cap \mathbb{R} \times \{a, b\}$ is a disjoint union of open intervals.

Put

$$\partial_- S = S \cap (\mathbb{R} \times \{a\}), \quad \partial_+ S = S \cap (\mathbb{R} \times \{b\}), \quad \partial S = \partial_- S \cup \partial_+ S.$$

A model strip $\mathbb{R} \times (a, b)$ will be called *open*.

Each model strip S admits a natural 1-dimensional foliation into parallel lines $\mathbb{R} \times t$ and boundary intervals from ∂S . We will call this foliation *canonical*. The following lemma implies that this foliation belongs to class \mathcal{F} .

Lemma 1.6. (e.g. [15]). Let $a < b \in \mathbb{R}$, $X = \mathbb{R}^2 \setminus ((-\infty, a] \cup [b, +\infty))$, and $\varepsilon > 0$. Then there exists a homeomorphism $\phi : \mathbb{R}^2 \rightarrow X$ such that

- (a) ϕ is fixed outside $\mathbb{R} \times (-\varepsilon, \varepsilon)$;
- (b) ϕ preserves foliations by horizontal lines, that is $\phi(\mathbb{R} \times t) = t \times \mathbb{R}$ for $t \neq 0$ and $\phi(\mathbb{R} \times 0) = (a, b) \times 0$, see Figure 1.3.

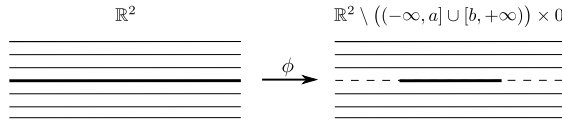


FIGURE 1.3.

Example 1.7. The foliation in Example 1.4 splits into five model strips such that

$$A \cong E \cong \mathbb{R} \times (0, 1) \bigcup (0, 1) \times 1,$$

$$B \cong C \cong D \cong \mathbb{R} \times (0, 1) \bigcup ((0, 1) \cup (2, 3)) \times 1.$$

Let $\mathbb{R} \times [-1, 1]$ be a model strip, $\phi_+, \phi_- : \mathbb{R} \times \{-1\} \rightarrow \mathbb{R} \times \{+1\}$ be two homeomorphisms given by

$$\phi_+(t, -1) = (t, 1), \quad \phi_-(t, -1) = (-t, 1),$$

for $t \in \mathbb{R}$, and $C = \mathbb{R} \times [-1, 1]/\phi_+$ and $M = \mathbb{R} \times [-1, 1]/\phi_-$ be the quotient spaces. Thus C (resp. M) is obtained from $\mathbb{R} \times [-1, 1]$ by identifying its boundary lines via preserving (resp. reversing) orientation homeomorphism. Therefore C is a cylinder and M is a Möbius band. Moreover, the canonical foliation on $\mathbb{R} \times [-1, 1]$ yields certain foliations Δ_C and Δ_M on C and M respectively also belonging to class \mathcal{F} . We will call C a *standard cylinder* and M a *standard Möbius band*.

Foliation associated with a regular function. A continuous function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ will be called *regular* whenever for each $z \in \mathbb{R}^2$ there are local coordinates (u, v) in which $z = (0, 0)$ and $f(u, v) = u + \text{const}$.

It follows that the partition Δ of \mathbb{R}^2 into connected components of level-sets $f^{-1}(t)$, $t \in \mathbb{R}$, of f is a *foliation* in a usual sense, i.e. it is *locally* homeomorphic with a partition of \mathbb{R}^2 into parallel lines. We will say that Δ is a *foliation associated with f* .

Notice that f has no local extremes, whence all leaves of f are homeomorphic with \mathbb{R} . Indeed, if Δ has a closed leaf ω , then by Jordan theorem ω bounds a 2-disk. Since f is constant on ω , it must have a local extreme inside that disk, which gives a contradiction.

Let $J \subset \mathbb{R}$ be a connected subset, i.e. either open or closed or half-closed interval. Then by a *cross-section* $\sigma : J \rightarrow \mathbb{R}^2$ of Δ we will mean a continuous path intersecting each

leaf at most once. It easily follows that σ is a cross-section if and only if the composition $f \circ \sigma : J \rightarrow \mathbb{R}$ is strictly monotone.

By a *saturation of a cross-section* $\sigma : J \rightarrow \mathbb{R}^2$ we will mean the saturation of its image $S(\sigma(J))$ and denote it simply by $S(\sigma)$, c.f. [13, §1.4]

Kaplan [13, Theorem 30] proved that for a cross-section $\sigma : [a, b] \rightarrow \mathbb{R}^2$ of Δ its saturation $S(\sigma)$ is foliated homeomorphic with $\mathbb{R} \times [a, b]$ foliated by parallel lines. However, this result can be misleading, since $S(\sigma)$ is not necessarily a closed subset of X .

For instance, consider the foliation in Figure 1.2(b). Let $\sigma : [a, b] \rightarrow \mathbb{R}^2$ be a cross-section passing through the special leaf α and such that $\sigma(a) \in A$ and $\sigma(b) \in B$. Then $\overline{S(\sigma)} \setminus S(\sigma) = \beta$.

Kaplan's construction. In [13, Theorem 29] W. Kaplan has shown that the foliation Δ associated with a regular function f belongs to class \mathcal{F} . In fact, he associated to Δ a family of pairs $\xi = \{(\omega_i, \sigma_i)\}_{i=-a}^b$ for some $a, b \in \mathbb{N} \cup \{\infty\}$, where

- (i) ω_i is a leaf being special for $i \neq 0$;
- (ii) $\sigma_0 : (-1, 1) \rightarrow \mathbb{R}^2$, $\sigma_i : [0, 1) \rightarrow \mathbb{R}^2$ for $i > 0$, and $\sigma_i : (-1, 0] \rightarrow \mathbb{R}^2$ for $i < 0$ are certain proper cross-sections of Δ ;
- (iii) $\sigma_i(0) \in \omega_i$ for all i ,

$$\begin{aligned} \sigma_i[0, 1) \cap S\left(\bigcup_{j=0}^{i-1} \sigma_j[0, 1)\right) &= \sigma_i(0), & i > 0, \\ \sigma_i(-1, 0] \cap S\left(\bigcup_{j=i+1}^0 \sigma_j(-1, 0]\right) &= \sigma_i(0), & i < 0. \end{aligned}$$

Kaplan proved that ξ determines Δ up to a foliated homeomorphism.

As noted above $S(\sigma_0)$ is foliated homeomorphic with $\mathbb{R} \times (0, 1)$ while $S(\sigma_i)$, $i \neq 0$, is foliated homeomorphic with a strip $\mathbb{R} \times [0, 1)$. Therefore the family ξ determines at most countable family of strips $\{V_i = S(\sigma_i)\}$ such that V_{i+1} is glued to V_i along the interval ω_i in their boundaries.

Kaplan's aim was to decrease the family of such strips as much as possible, see first paragraph of [13, Section 3.1]. However, the construction of family ξ then becomes ambiguous and depends on a particular choice of special leaves and cross-sections. This is illustrated in Figure 1.4(b), where two such families for the same foliation are presented.

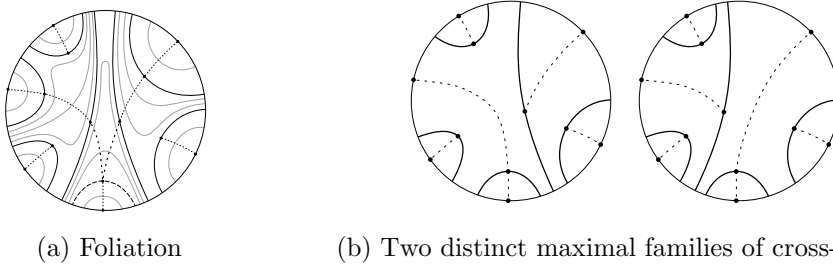


FIGURE 1.4.

On the other hand cutting \mathbb{R}^2 along special leaves is an unambiguous procedure and it gives a canonical decomposition of \mathbb{R}^2 .

In the present paper we extend Kaplan's results to foliations Δ from class \mathcal{F} on arbitrary surfaces X and describe the topological structure of connected components of $X \setminus \Sigma$ and their closures, where Σ is the union of all special leaves of Δ .

Theorem 1.8. *Let X be a connected 2-dimensional manifold and Δ be a foliation on X belonging to class \mathcal{F} . Suppose that the family Σ of all special leaves of Δ is locally finite,*

and let Q be a connected component of $X \setminus (\Sigma \cup \partial X)$. Then the following statements hold true.

- (1) Q is foliated homeomorphic either with a standard cylinder C or a standard Möbius band M or an open model strip $\mathbb{R} \times (-1, 1)$. Moreover, in the first two cases $Q = X$.
- (2) Suppose Q is foliated homeomorphic with an open model strip. Fix any foliated homeomorphism $\phi : \mathbb{R} \times (-1, 1) \rightarrow Q$ and denote

$$A = \phi(\mathbb{R} \times (-1, 0]), \quad B = \phi(\mathbb{R} \times [0, 1)).$$

Then the closures \overline{A} and \overline{B} are foliated homeomorphic to some model strips.

This theorem implies that the topological structure of the foliation $\Delta \in \mathcal{F}$ is uniquely determined by the combinatorics of gluing model strips. Also notice that the intersection $(\overline{A} \cap \overline{B}) \setminus \phi(\mathbb{R} \times 0)$ can be non-empty, whence one can not expect that $\overline{Q} = \overline{A} \cup \overline{B}$ is homeomorphic with a model strip.

The proof of Theorem 1.8 will be given in §4 and 6.

2. SPECIAL POINTS OF NON-HAUSDORFF SPACES

Throughout this section Y be a topological space.

Definition 2.1. Let $y \in Y$ and β_y be the family of all neighborhoods of y . Then the following set

$$\text{hcl}(y) := \bigcap_{V \in \beta_y} \overline{V}$$

will be called the *Hausdorff closure* of y . We will say that y is a *special* point of Y whenever $y \neq \text{hcl}(y)$. The set of all special points of Y will be denoted by \mathcal{V} .

Notice that Y is Hausdorff if and only if $y = \text{hcl}(y)$ for all $y \in Y$, i.e. when $\mathcal{V} = \emptyset$.

- Lemma 2.2.**
- (1) Let $y, z \in Y$. Then $y \in \text{hcl}(z)$ if and only if $z \in \text{hcl}(y)$, however, in general, $\text{hcl}(y) \neq \text{hcl}(z)$.
 - (2) Let $f : Y \rightarrow Z$ be a continuous map into a Hausdorff topological space Z . Then $f(\text{hcl}(y)) = f(y)$ for all $y \in Y$.
 - (3) The set $Y \setminus \mathcal{V}$ of all non-special points is Hausdorff.

Proof. (1) Suppose $y \in \text{hcl}(z) = \bigcap_{V \in \beta_z} \overline{V}$, that is y belongs to the closure of each neighborhood of z which means in turn that every neighborhood of y intersect every neighborhood of z . The latter property is symmetric with respect to y and z , whence $z \in \text{hcl}(y)$ as well.

(2) Suppose $z \in \text{hcl}(y)$ but $f(y) \neq f(z)$. Since Z is Hausdorff, there exist open disjoint neighborhoods $W_{f(y)}$ and $W_{f(z)}$ of points $f(y)$ and $f(z)$. But then their inverses $V_y = f^{-1}(W_{f(y)})$ and $V_z = f^{-1}(W_{f(z)})$ are disjoint open neighborhoods of y and z respectively. Hence z can not belong to $\text{hcl}(y)$ which contradicts to the assumption.

(3) Let $y, z \in Y \setminus \mathcal{V}$ be two distinct points. Thus $y = \text{hcl}(y) \neq z$ and so there exist disjoint neighborhoods V_y and V_z of y and z respectively. This implies that $Y \setminus \mathcal{V}$ is Hausdorff. \square

Non-Hausdorff one-dimensional manifolds. Let Y be a T_1 -topological space locally homeomorphic with open sets of $[0, 1)$. Notice that we allow Y to be non-Hausdorff. Then as usual the set of points having an open neighborhood homeomorphic with $(0, 1)$ will be denoted by $\text{Int } Y$ and called the *interior* of Y , while its complement $\partial Y := Y \setminus \text{Int } Y$ will be called the *boundary* of Y .

Lemma 2.3. *Suppose that the set \mathcal{V} of special points of Y is locally finite. Then every connected component W of $Y \setminus \mathcal{V}$ is open in Y and is homeomorphic with one of the following spaces: $[0, 1)$, $(0, 1)$, $[0, 1]$, S^1 . In the last two cases, i.e. when W is compact, W is a connected component of Y .*

Every connected component of $Y \setminus (\mathcal{V} \cup \partial Y)$ is homeomorphic with $(0, 1)$.

Proof. Since Y is a T_1 -space, every point $y \in Y$ is a closed subset. Also since \mathcal{V} is locally finite, it follows that \mathcal{V} is a closed subset, whence by (3) of Lemma 2.2 $Y \setminus \mathcal{V}$ is a Hausdorff topological space locally homeomorphic with $[0, 1)$. Hence every connected component W of $Y \setminus \mathcal{V}$ is a one-dimensional manifold and so it is homeomorphic with one of the spaces $[0, 1)$, $(0, 1)$, $[0, 1]$, S^1 . Moreover, since $Y \setminus \mathcal{V}$ is locally connected, we obtain that W is open in $Y \setminus \mathcal{V}$ and therefore in Y as well.

Suppose W is compact, i.e. it is homeomorphic either with $[0, 1]$ or with S^1 . Let us show that then W is also closed in Y . This will imply that W is a connected component of Y .

Let $\{y_i\}_{i \in \mathbb{N}} \subset W$ be a sequence converging to some $z \in Y$. We should prove that $z \in W$. Since W is compact, that sequence also converges to some $y \in W$. Hence if V_y and V_z are any two open neighborhoods of y and z respectively, then there exists $n > 0$ such that $y_n \in V_y \cap V_z$. Thus $V_y \cap V_z \neq \emptyset$, which implies that $z \in \text{hcl}(y) = \{y\}$, that is $z = y \in W$.

We leave the last statement for the reader. \square

Suppose Y is connected and not homeomorphic with a circle. Let $\{W_\alpha\}_{\alpha \in \mathcal{A}}$ be the family of all connected components of $\mathcal{V} \cup \partial Y$. Then due to Lemma 2.3 for each $\alpha \in \mathcal{A}$ there exists a homeomorphism $\phi_\alpha : (-1, 1) \rightarrow W_\alpha$. Consider the following collection of subsets:

$$\mathcal{B} = \left\{ \phi_\alpha(-1, -\tfrac{1}{2}], \phi_\alpha[\tfrac{1}{2}, 1) \right\}_{\alpha \in \mathcal{A}}.$$

Let $K \in \mathcal{B}$. Then we denote $K^\circ := \phi_\alpha(-1, -\frac{1}{2})$ if $K = \phi_\alpha(-1, -\frac{1}{2}]$, and $K^\circ := \phi_\alpha(\frac{1}{2}, 1)$ if $K = \phi_\alpha[\frac{1}{2}, 1)$ for some $\alpha \in \mathcal{A}$. Thus each $K \in \mathcal{B}$ is homeomorphic with a half-open segment $[0, 1)$, and K° is the subset of K corresponding to $(0, 1)$.

Lemma 2.4. *Let $y \in \partial Y$. Then there exists a unique $K \in \mathcal{B}$ such that $y \in \overline{K}$. In this case $V := \{y\} \cup K^\circ$ is an open neighborhood of y and there exists a homeomorphism $\psi : [0, 1) \rightarrow \{y\} \cup K$ such that $\psi(0) = y$ and $\psi(0, 1) = V$.*

Suppose $y \in \mathcal{V} \setminus \partial Y$. Then there exist two distinct elements $K, L \in \mathcal{B}$ such that $y \in \overline{K} \cap \overline{L}$ and $y \notin \overline{M}$ for all other $M \in \mathcal{B}$. Moreover, the set $V := K^\circ \cup \{y\} \cup L^\circ$ is an open neighborhood of y and there exists a homeomorphism $\mu : [-1, 1] \rightarrow K \cup \{y\} \cup L$ such that $\psi(0) = y$ and $\psi(-1, 1) = V$.

Proof. We will consider only the case $y \in \mathcal{V} \setminus \partial Y$. Notice that the family $\{\phi_\alpha[-1, 1]\}_{\alpha \in \mathcal{A}}$ is locally finite and consists of closed sets. Therefore its union $Z = \bigcup_{\alpha \in \mathcal{A}} \phi_\alpha[-1, 1]$ is closed. Hence the set $T = (\mathcal{V} \setminus \{y\}) \cup \partial Y \cup Z$ is closed and does not contain y . Therefore there exists a neighborhood $J \subset Y \setminus T$ of y and a homeomorphism $\mu : [-\frac{1}{2}, \frac{1}{2}] \rightarrow J$ such that $\mu(0) = y$ and $\mu(-\frac{1}{2}, \frac{1}{2})$ is an open neighborhood of y .

Notice that $J \setminus \{y\}$ consists of exactly two connected components $A = \mu(-\frac{1}{2}, 0)$ and $B = \mu(0, \frac{1}{2}]$, and is contained in $Y \setminus (\mathcal{V} \cup \partial Y \cup Z) = \bigcup_{K \in \mathcal{B}} K^\circ$. Hence $A \subset K^\circ$ and $B \subset L^\circ$ for some $K, L \in \mathcal{B}$, see Figure 2.1.

Moreover, any other neighborhood of y intersects both A and B and therefore both K and L . Hence $y \in \overline{K} \cap \overline{L}$ and $y \notin \overline{M}$ for all other $M \in \mathcal{B}$ distinct from K and L .

Fix any homeomorphisms $\kappa : [-1, 0) \rightarrow K$ and $\lambda : (0, 1] \rightarrow L$. Notice that A is not contained in any compact subset P of K , since otherwise $y \in \overline{A} \subset P \subset K$, which

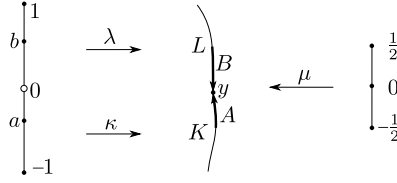


FIGURE 2.1.

contradicts to the assumption that $y \notin K$. This implies that $\kappa^{-1}(A) = [a, 0) \subset (-1, 0)$, where $a = \kappa^{-1} \circ \mu(-\frac{1}{2}) \in (-1, 0)$. By the same arguments, $\lambda^{-1}(B) = (0, b] \subset (0, 1)$, where $b = \lambda^{-1} \circ \mu(\frac{1}{2}) \in (0, 1)$.

Lemma 2.4.1. $K \neq L$.

Proof. If $K = L$, then we have a homeomorphism $\mu = \lambda^{-1} \circ \kappa : [-1, 0) \rightarrow (0, 1]$. Hence there exists $c \in (a, 0)$ such that $c' = \mu(c) \in (0, b)$. Then $\kappa(c) = A$ and $\lambda \circ \mu(c) \in B$. But $\lambda \circ \mu(c) = \kappa(c)$, and so $A \cap B \neq \emptyset$, which contradicts to the assumption. \square

Now fix arbitrary orientation preserving homeomorphisms $\eta_K : [-1, -\frac{1}{2}] \rightarrow [-1, a]$ and $\eta_L : [\frac{1}{2}, 1] \rightarrow [b, 1]$ and define the map $\psi : [-1, 1] \rightarrow K \cup \{y\} \cup L$ by the formula

$$\psi(t) = \begin{cases} \kappa^{-1} \circ \eta_K(t), & t \in [-1, -\frac{1}{2}], \\ \mu(t), & t \in [-\frac{1}{2}, \frac{1}{2}], \\ \lambda^{-1} \circ \eta_L(t), & t \in [\frac{1}{2}, 1]. \end{cases}$$

One easily checks that ψ is a required homeomorphism. \square

3. PARTITIONS

Let X be a topological space, Δ be a partition of X , $Y = X/\Delta$ be the quotient space, and $p : X \rightarrow Y$ be the corresponding quotient map. We will endow Y with the *factor topology*, so a subset $V \subset Y$ is open if and only if its inverse $p^{-1}(V)$ is open in X .

A *saturation* $S(U)$ of a subset $U \subset X$ with respect to Δ is the union of all $\omega \in \Delta$ such that $\omega \cap U \neq \emptyset$. Equivalently, $S(U) = p^{-1}(p(U))$. A subset U is *saturated* if $U = S(U)$. Evidently, if $A \cap S(B) = \emptyset$, then $S(A) \cap S(B) = \emptyset$ as well.

Lemma 3.1. (1) Y is a T_1 -space if and only if each element $\omega \in \Delta$ is closed.

(2) The following conditions are equivalent:

- (a) the map $p : X \rightarrow Y$ is open;
- (b) for each $x \in X$ there exists an open neighborhood U whose saturation $S(U)$ is open;
- (c) there exists an open cover $\beta = \{U_i\}_{i \in \Lambda}$ of X such that for each $i \in \Lambda$ the restriction $p|_{U_i} : U_i \rightarrow p(U_i)$ is an open map.

(3) If p is open then for each saturated subset B we have that

$$(3.1) \quad X \setminus \overline{B} = S(X \setminus \overline{B}),$$

$$(3.2) \quad p(\overline{B}) = \overline{p(B)}.$$

In particular, $\overline{S(A)}$ and $X \setminus \overline{S(A)}$ are saturated for each subset $A \subset X$.

- (4) Let $\beta = \{W_i\}_{i \in \Lambda}$ be a family of subsets of Y , and $\alpha = \{p^{-1}(W_i)\}_{i \in \Lambda}$ be the corresponding family of their inverses in X . If β is locally finite, then so is α . Conversely, if α is locally finite and p is open then β is locally finite as well.
- (5) Suppose X is a normal topological space and $\alpha = \{\omega_i\}_{i \in \mathbb{N}}$ is a locally finite family of mutually disjoint closed subsets of X . Then for each $i \in \mathbb{N}$ there exists a neighborhood U_i of ω_i such that $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$.

- (6) Let $f : A \rightarrow B$ be a bijection between topological spaces. Suppose that $\{K_i\}_{i \in \Lambda}$ is a locally finite cover of A by closed sets. If each of the restrictions $f|_{K_i} : K_i \rightarrow B$ is continuous, then f is continuous itself.

Moreover, suppose the family $\{\psi(K_i)\}_{i \in \Lambda}$ is locally finite, $f(K_i)$ is closed in B , and the restriction $f|_{K_i} : K_i \rightarrow f(K_i)$ is a homeomorphism for each $i \in \Lambda$. Then f is a homeomorphism.

Proof. Statements (1), (2), and (6) are easy and we leave them for the reader.

(3) Suppose p is an open map and let $B \subset X$ be a saturated subset. Then $X \setminus B$ is also saturated, i.e. $S(X \setminus B) = X \setminus B$, and so

$$X \setminus \overline{B} \subset S(X \setminus \overline{B}) \subset S(X \setminus B) = X \setminus B.$$

Hence

$$\overline{B} \supset X \setminus S(X \setminus \overline{B}) \supset B.$$

As $X \setminus \overline{B}$ is open, $S(X \setminus \overline{B})$ is open as well, and therefore $X \setminus S(X \setminus \overline{B})$ is a closed subset containing B . Therefore it must contain the closure \overline{B} , hence $\overline{B} = X \setminus S(X \setminus \overline{B})$, which implies (3.1).

Let us prove (3.2). Since p is continuous, $p^{-1}(\overline{p(B)})$ is a closed subset containing B . Therefore it contains \overline{B} , and so $p(\overline{B}) \subset \overline{p(B)}$.

Conversely, by (3.1), \overline{B} is saturated and closed. Therefore, by definition of the quotient topology, $p(\overline{B})$ is a closed subset and it contains $p(B)$. Hence it also contains $\overline{p(B)}$, i.e. $p(\overline{B}) \supset \overline{p(B)}$.

(4) Suppose β is a locally finite family and $x \in X$. We should find a neighborhood U of x which intersects only finitely many elements from α . Let $y = p(x)$. Since β is locally finite, there exists a neighborhood V of y intersecting only finitely many elements $W_{i_1}, \dots, W_{i_k} \in \beta$. Then $p^{-1}(V)$ is an open neighborhood of x intersecting only the following elements $p^{-1}(W_{i_1}), \dots, p^{-1}(W_{i_k})$ of α .

Conversely, suppose α is locally finite and p is open. Let $y \in Y$ and $x \in X$ be such that $p(x) = y$. Then there exists a neighborhood U of x intersecting only finitely many elements, say $p^{-1}(W_{i_1}), \dots, p^{-1}(W_{i_k})$, of α . Therefore its saturation $S(U) = p^{-1}(p(U))$ also intersects only $p^{-1}(W_{i_1}), \dots, p^{-1}(W_{i_k})$.

Since p is open, the image $p(U)$ is an open neighborhood of y . We claim that $p(U)$ intersects only the elements W_{i_1}, \dots, W_{i_k} of α . Indeed, if $p(U) \cap W_i \neq \emptyset$ for some $i \in \Lambda$, then $p^{-1}(p(U)) \cap p^{-1}(W_i) \neq \emptyset$ which is possible only when $i \in \{i_1, \dots, i_k\}$.

(5) For each $i \in \mathbb{N}$ consider the following subfamily $\alpha_i = \{\omega_j\}_{j \geq i}$ of α , so $\alpha = \alpha_1$ and $\alpha_{i+1} \subset \alpha_i$ for all $i \in \mathbb{N}$. Then each α_i is locally finite as well, and therefore the union $A_i = \bigcup_{j=i}^{\infty} \omega_j$ is a closed subset of X .

Since X is normal and ω_1 and A_2 are mutually disjoint and closed, there exists an open neighborhood U_1 of ω_1 such that $\overline{U_1} \cap A_2 = \emptyset$. Then ω_2 and $\overline{U_1} \cup A_3$ are mutually disjoint and closed, whence there exists an open neighborhood U_2 of ω_2 which does not intersect $\overline{U_1} \cup A_3$. Repeating these arguments so on we will construct for each $i \in \mathbb{N}$ an open neighborhood U_i of ω_i such that $\overline{U_i}$ does not intersect $(\bigcup_{j=1}^{i-1} \overline{U_j}) \cup A_{i+1}$. Then $\overline{U_i} \cap \overline{U_j} = \emptyset$ for all $i \neq j \in \mathbb{N}$. \square

Definition 3.2. We will say that a partition Δ is *locally trivial* if for each $\omega \in \Delta$ there exists an open neighborhood U , a topological space J , a point $t_0 \in J$, and a homeomorphism $\phi : \omega \times J \rightarrow U$ such that $\phi(\omega \times t)$ is an element of Δ for all $t \in J$ and $\phi(x, t_0) = x$ for all $x \in \omega$.

In particular, a foliation belonging to class \mathcal{F} is a locally trivial partition.

Notice that in the notation of Definition 3.2 U is saturated and open in X , whence its image $V = p(U)$ is open in Y and we have the following commutative diagram:

$$(3.3) \quad \begin{array}{ccc} \omega \times J & \xrightarrow[\cong]{\phi} & U = p^{-1}(V) \\ q_2 \downarrow & & \downarrow p \\ J & \xrightarrow[1-1]{\xi} & V \end{array}$$

where q_2 is a projection onto the second multiple and ξ is the induced one-to-one continuous map but it is not necessarily a homeomorphism.

Lemma 3.3. *The following conditions are equivalent:*

- (1) *the quotient map $p : X \rightarrow Y$ is a locally trivial fibration;*
- (2) *partition Δ is locally trivial and the quotient map $p : X \rightarrow Y$ is open.*

Proof. (1) \Rightarrow (2). Suppose p is a locally trivial fibration. We claim that then Δ is locally trivial. Indeed, let $\omega \in \Delta$ and $y = p(\omega) \in Y$. Since p is locally trivial, there exists a neighborhood V of y and the following commutative diagram:

$$(3.4) \quad \begin{array}{ccc} \omega \times V & \xrightarrow[\cong]{\phi} & U_\omega = p^{-1}(V) \\ q_2 \downarrow & & \downarrow p \\ V & \xrightarrow[\xi = \text{id}_V]{} & V \end{array}$$

in which ϕ is a homeomorphism. This diagram coincides with (3.3) for $J = V$, and therefore Δ is a locally trivial partition.

Let us prove that p is an open map. Notice that in Diagram (3.4) q_2 is an open map as a coordinate projection. Since ϕ is a homeomorphism, it follows that the restriction $p|_{U_\omega}$ is an open map as well. But then $\beta = \{U_\omega\}_{\omega \in \Delta}$ is an open cover of X such that each restriction $p|_{U_\omega}$ is open. Therefore by (2) of Lemma 3.1 p is open.

(2) \Rightarrow (1). Suppose p is an open map and Δ is locally trivial. We claim that then in (3.3) the map ξ is open, and therefore it is a homeomorphism. This will imply that p is a locally trivial fibration.

Let $T \subset J$ be an open subset. Then $\phi \circ q_2^{-1}(T)$ is open in U . Since p is open, we get that $\xi(T) = p \circ \phi \circ q_2^{-1}(T)$ is open in V . Thus ξ is an open map. \square

Definition 3.4. An element $\omega \in \Delta$ will be called *special* if its image $y = p(\omega) \in Y$ is a special point of Y , i.e. $y \neq \text{hcl}(y) := \bigcap_{V \in \beta_y} \overline{V}$, where β_y is the family of all neighborhoods of y , see Definition 2.1. Let also

$$\text{hcl}(\omega) = \bigcap_{N(\omega)} \overline{S(N(\omega))}, \quad \text{hcl}_S(\omega) = \bigcap_{N_S(\omega)} \overline{N_S(\omega)},$$

where $N(\omega)$ runs over *all open* neighborhoods of ω and $N_S(\omega)$ runs over *all saturated open* neighborhoods of ω .

Lemma 3.5. *Let $\omega \in \Delta$ and $y = p(\omega)$. Then*

$$\text{hcl}(\omega) \subset \text{hcl}_S(\omega) \subset p^{-1}(\text{hcl}(y)).$$

If p is an open map, then

$$\text{hcl}(\omega) = \text{hcl}_S(\omega) = p^{-1}(\text{hcl}(y)), \quad p(\text{hcl}_S(\omega)) = \text{hcl}(y).$$

Proof. First we establish relations between $\text{hcl}(\omega)$ and $\text{hcl}_S(\omega)$. Notice that the family $\mathcal{A} = \{S(N(\omega))\}$ of saturations of all open neighborhoods of ω includes the family $\mathcal{B} = \{N_S(\omega)\}$ of all saturated open neighborhoods of ω . Therefore the intersection $\text{hcl}(\omega)$ of the larger family \mathcal{A} is contained in the intersection $\text{hcl}_S(\omega)$ of the smaller family \mathcal{B} , that is $\text{hcl}(\omega) \subset \text{hcl}_S(\omega)$.

If p is an open map, so the saturation of an open set is open, then $\mathcal{A} = \mathcal{B}$, and therefore $\text{hcl}(\omega) = \text{hcl}_S(\omega)$.

Now we will describe relationships between $\text{hcl}_S(\omega)$ and $\text{hcl}(y)$. By definition of the quotient topology on Y the map p induces a bijection between the families \mathcal{B} and β_y . Moreover, if $N_S(\omega) \in \mathcal{B}$ is an open saturated neighborhood of ω and $V = p(N_S(\omega))$ is an open neighborhood of y , then, due to continuity of p , we have that $p(\overline{N_S(\omega)}) \subset \overline{V}$. Hence $p(\text{hcl}_S(\omega)) \subset \text{hcl}(y)$, that is $\text{hcl}_S(\omega) \subset p^{-1}(\text{hcl}(y))$.

If p is open, then, due to (3.2), $p(\overline{N_S(\omega)}) = \overline{p(N_S(\omega))} = \overline{V}$, whence

$$(3.5) \quad p(\text{hcl}_S(\omega)) = p\left(\bigcap_{N_S(\omega) \in \mathcal{B}} \overline{N_S(\omega)}\right) = p\left(\bigcap_{V \in \beta_y} p^{-1}(\overline{V})\right) = \bigcap_{V \in \beta_y} \overline{V} = \text{hcl}(y).$$

Finally, as $\text{hcl}_S(\omega)$ is saturated as an intersection of saturated sets, it follows from (3.5) that $\text{hcl}_S(\omega) = p^{-1}(\text{hcl}(y))$. \square

Lemma 3.6. *Suppose that the following conditions hold true:*

- (a) $p : X \rightarrow Y$ is a locally trivial fibration with fiber \mathbb{R} ;
- (b) the set Σ of special elements of X is locally finite;
- (c) Y is a T_1 -space locally homeomorphic with open subsets of $[0, 1)$.

Then every connected component Q of $X \setminus \Sigma$ is open in X and is foliated homeomorphic with one of the following five stripped surfaces: model strips $\mathbb{R} \times (0, 1)$, $\mathbb{R} \times [0, 1)$, $\mathbb{R} \times [0, 1]$, or standard cylinder C , or standard Möbius band M . Moreover, in the last three cases, Q is also closed in X .

Proof. By (a) and Lemma 3.3 p is an open map. Therefore by Lemma 3.5 $\Sigma = p^{-1}(\mathcal{V})$ and $p(\Sigma) = \mathcal{V}$, where \mathcal{V} is the set of special points of Y . Then by (b) and (4) of Lemma 3.1 \mathcal{V} is also a locally finite family of points. Due to (c) each point in Y is closed, whence \mathcal{V} is closed in Y .

Let W be a connected component $Y \setminus \mathcal{V}$. Then W is open in Y and open closed in $Y \setminus \mathcal{V}$. Therefore $Q = p^{-1}(W)$ is open in X and open closed in $X \setminus \Sigma$, i.e. Q is a connected component of $X \setminus \Sigma$. Moreover, due to (a) the restriction $p : Q \rightarrow W$ is a locally trivial fibration with fiber \mathbb{R} , and by Lemma 2.3 W is homeomorphic with one of the following spaces: $(0, 1)$, $[0, 1)$, $[0, 1]$, S^1 . Therefore in the first three cases (when W is contractible) Q is fiber-wise homeomorphic to a product $\mathbb{R} \times W$, and in the last case, when $W \cong S^1$, Q is fiber-wise homeomorphic either with the standard cylinder C or with the standard Möbius band M .

It remains to show that every connected component Q of $X \setminus \Sigma$ can be represented as $Q = p^{-1}(W)$ for some connected component W of $Y \setminus \mathcal{V}$. Let $W = p(Q)$. We claim that W is open closed in $Y \setminus \mathcal{V}$. Indeed, let W' be the connected component of $Y \setminus \mathcal{V}$ containing W . Then as noted above $p^{-1}(W')$ is connected and contains Q , whence $Q = p^{-1}(W')$, and so $W = W'$. \square

4. PROOF OF (1) OF THEOREM 1.8

Let X be a 2-dimensional manifold and Δ be a 1-dimensional foliation on X belonging to class \mathcal{F} and such that the set Σ of special leaves of X is locally finite. Let also $Y = X/\Delta$ be the space of leaves endowed with the corresponding factor topology and $p : X \rightarrow Y$ be the factor map.

We claim that p satisfies conditions (a)–(c) of Lemma 3.6. Indeed, by Lemma 1.2 p is open, and by Lemma 3.3 it is a locally trivial fibration with fiber \mathbb{R} , so condition (a) holds. Condition (b) holds by assumption and condition (c) directly follows from definition of class \mathcal{F} .

Therefore by Lemma 3.6 every connected component $X \setminus \Sigma$ is foliated homeomorphic with one of the spaces: $\mathbb{R} \times (0, 1)$, $\mathbb{R} \times [0, 1)$, $\mathbb{R} \times [0, 1]$, C , M .

Applying the above result to the surface $X \setminus \partial X$ we get that every connected component of $X \setminus (\Sigma \cup \partial X)$ is foliated homeomorphic with one of the spaces: $\mathbb{R} \times (0, 1)$, C , or M . Statement (1) of Theorem 1.8 is proved.

5. TRAPEZOIDS

The results of this section will be used for the proof of (2) of Theorem 1.8.

Let $c < d$ and $\alpha, \beta : (c, d] \rightarrow \mathbb{R}$ be two continuous functions such that $\alpha(y) < \beta(y)$ for all $y \in (c, d]$. Then the subset

$$T = \{(x, y) \in \mathbb{R}^2 \mid \alpha(y) \leq x \leq \beta(y), c < y \leq d\}$$

will be called a *half open trapezoid* or simply a *trapezoid*. In this case $[\alpha(d), \beta(d)] \times d$ is the *upper base* of T , d is the *level* of the upper base, $d - c$ is the *altitude* of T , and the set

$$\text{roof}(T) := \{(\alpha(y), y)\}_{y \in (c, d]} \cup [\alpha(d), \beta(d)] \times d \cup \{(\beta(y), y)\}_{y \in (c, d]}$$

is the *roof* of T , see Figure 5.1a).

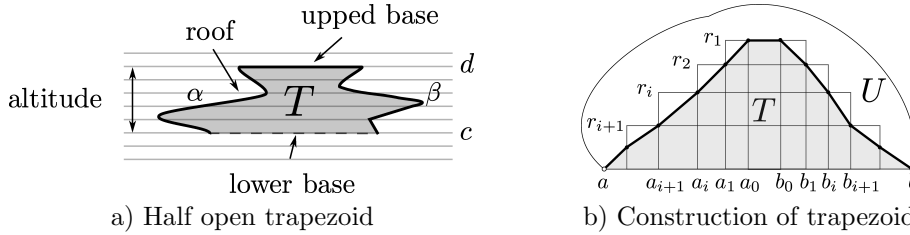


FIGURE 5.1. Half open trapezoid

Notice that if $\phi : \mathbb{R} \times (c, d] \rightarrow \mathbb{R} \times (c, d]$ is a homeomorphism preserving second coordinate, i.e. $\phi(\mathbb{R} \times y) = \mathbb{R} \times y$ for all $y \in (c, d]$, then $\phi(T)$ is a trapezoid as well.

In general, α and β can be non-bounded or have no limits when $y \rightarrow c + 0$. Suppose, in addition, that there exist finite or infinite limits

$$\lim_{y \rightarrow c+0} \alpha(y) = a, \quad \lim_{y \rightarrow c+0} \beta(y) = b$$

such that $a < b$. Then $(a, b) \times c$ will be called the (*lower*) *base* of T . If a and b are finite numbers, then T will be called a *trapezoid with bounded base*, and the set

$$\overline{T} = T \cup [a, b] \times c$$

will be a *closed* trapezoid. In particular, if α and β are constant functions, then the trapezoid T will be called a *rectangle*.

Lemma 5.1. *Let $J = (a, b) \times 0 \subset \mathbb{R}^2$ be an open interval, $N = J \cup \mathbb{R}^2 \times (0, +\infty)$, and U be an open neighborhood of J in N . Then there exists a half open trapezoid $T \subset U$ with base J , see Figure 5.1b).*

Proof. Fix any two sequences $\{a_i\}_{i=0}^\infty, \{b_i\}_{i=0}^\infty \subset (a, b)$ such that $\lim_{i \rightarrow \infty} a_i = a$, $\lim_{i \rightarrow \infty} b_i = b$, and

$$\cdots < a_{i+1} < a_i < \cdots < a_0 < b_0 < \cdots < b_i < b_{i+1} < \cdots$$

Let also $J_i = [a_i, b_i] \times 0$. Since U is an open neighborhood of J_i and J_i is compact, there exist $r_i > 0$ such that $J_i \times [0, r_i] \subset U$. One can assume that $\lim_{i \rightarrow \infty} r_i = 0$ and $\{r_i\}$ is strictly decreasing. Now let $\alpha, \beta : (0, d] \rightarrow (0, +\infty)$ be a unique continuous function such that for each $i \geq 0$

- (i) $\alpha(a_i) = \beta(b_i) = r_{i+1}$;
- (ii) the restrictions $\alpha|_{[a_{i+1}, a_i]}$ and $\beta|_{[b_i, b_{i+1}]}$ are linear.

Then one easily checks that the function α and β are strictly monotone and their inverses α^{-1} and β^{-1} determine a half open trapezoid $T \subset U$ with base J . \square

Proposition 5.2. *Let $T_i \subset \mathbb{R} \times (c, d]$, $i \in \mathbb{N}$, be a half open trapezoid with upper base at level $d_i \in (c, d]$ such that $T_i \cap T_j = \emptyset$ for $i \neq j$ and $\lim_{i \rightarrow \infty} d_i = c$. Then there exists a homeomorphism $\eta : \mathbb{R} \times (c, d] \rightarrow \mathbb{R} \times (c, d]$ such that*

- (i) $\eta(\mathbb{R} \times y) = \mathbb{R} \times y$ for all $y \in (c, d]$;
- (ii) $\eta(T_i)$ is a half open rectangle.

Proof. We need the following three lemmas. It will be convenient to say that for a function $f : [a, b] \rightarrow \mathbb{R}$ its *graph* is the subset $\{(f(y), y) \mid y \in [a, b]\} \subset \mathbb{R}^2$, so we just swap coordinates with respect to the usual definition. In particular, for $q \in \mathbb{R}$ a vertical segment $q \times [a, b]$ can be regarded as a graph of a constant function $[a, b] \mapsto q$.

Lemma 5.2.1. (c.f. [14, Lemma 6.1.1]). *Let $\Delta_k = \{(y_1, \dots, y_k) \in \mathbb{R}^k \mid y_1 < y_2 < \dots < y_k\}$ and $q_1 < q_2 < \dots < q_k \in \mathbb{R}$. Then there exists a C^∞ function $u_k : \mathbb{R} \times \Delta_k \rightarrow \mathbb{R}$ having the following properties:*

- (a) *the correspondence $x \mapsto u_k(x; y_1, \dots, y_k)$ is an orientation preserving homeomorphism $\mathbb{R} \rightarrow \mathbb{R}$ for all $(y_1, \dots, y_k) \in \Delta_k$;*
- (b) *$u_k(x; q_1, \dots, q_k) = x$ for all $x \in \mathbb{R}$;*
- (c) *$u_k(y_i; y_1, \dots, y_k) = q_i$ for $i = 1, \dots, k$.*

Proof. The construction of u_k is similar to [14, Lemma 6.1.1]. For instance, one can set

$$u_1(x; y_1) = x - y_1 + q_1, \quad u_2(x; y_1, y_2) = q_1 + \frac{q_2 - q_1}{y_2 - y_1}(x - y_1).$$

We leave the details for the reader. \square

Lemma 5.2.2. *Let $\gamma_i : (c, s] \rightarrow \mathbb{R}$, $i = 1, \dots, k$, be a finite family of continuous functions such that $\gamma_i(y) \neq \gamma_j(y)$ whenever $i \neq j$ and $y \in (c, s]$. Then there exists a homeomorphism $\phi : \mathbb{R} \times (c, s] \rightarrow \mathbb{R} \times (c, s]$ such that*

- (1) $\phi(\mathbb{R} \times y) = \mathbb{R} \times y$ for all $y \in (c, s]$;
- (2) ϕ is fixed on $\mathbb{R} \times s$;
- (3) ϕ maps the graph $\{(\gamma_i(y), y) \mid y \in (c, s]\}$ of γ_i , $i \in \{1, \dots, k\}$, onto a vertical segment $\gamma_i(s) \times [a, s]$.

Proof. One can assume that $\gamma_i < \gamma_j$ for $i < j$. Let $u_k : \mathbb{R} \times \Delta_k \rightarrow \mathbb{R}$ be a function from Lemma 5.2.1 constructed for the numbers $q_i = \gamma_i(s)$, $i \in \{1, \dots, k\}$. Then a homeomorphism ϕ satisfying (1)-(3) can be defined by the following formula:

$$\phi(x, y) = (u_k(x; \gamma_1(y), \dots, \gamma_k(y)), y).$$

Indeed, (1) is evident. Moreover, due to property (b) of u_k we have that

$$\phi(x, s) = (u_k(x; \gamma_1(s), \dots, \gamma_k(s)), s) = (x, s)$$

which proves (2). Finally, by property (c) of u_k

$$\phi(\gamma_i(y), y) = (u_k(\gamma_i(y); \gamma_1(y), \dots, \gamma_k(y)), y) = (\gamma_i(s), y),$$

so (3) is also satisfied. \square

Lemma 5.2.3. *Let $\{d_i\}_{i \in \mathbb{N}} \subset (c, d]$ be a sequence with $\lim_{i \rightarrow \infty} d_i = c$, and for each $i \in \mathbb{N}$ let $\gamma_i : (c, d_i] \rightarrow \mathbb{R}$ be a continuous function such that the graphs of γ_i and γ_j are mutually disjoint for $i \neq j$. Then there exists a homeomorphism $\eta : \mathbb{R} \times (c, d] \rightarrow \mathbb{R} \times (c, d]$ such that*

- (i) $\eta(\mathbb{R} \times y) = \mathbb{R} \times y$ for all $y \in (c, d]$;
- (ii) η maps the graph $\{(\gamma_i(y), y) \mid y \in (c, d_i]\}$ of γ_i onto a vertical segment $q_i \times (c, d_i]$ for some $q_i \in \mathbb{R}$, $i \in \mathbb{N}$.

Proof. One can assume, in addition, that $\{d_i\}_{i \in \mathbb{N}}$ is non-increasing. Let us remove repeating elements from $\{d_i\}_{i \in \mathbb{N}}$ and denote the obtained sequence by $\{s_i\}_{i \in \mathbb{N}}$. Thus there is an increasing sequence of indices $1 = j_1 < j_2 < \dots < j_n < \dots$ such that

$$s_i = d_{j_i} = d_{j_i+1} = \dots = d_{j_{i+1}-1} > s_{i+1} = d_{j_{i+1}} = \dots.$$

Then by Lemma 5.2.2 there exists a homeomorphism $\phi_1 : \mathbb{R} \times (c, s_1] \rightarrow \mathbb{R} \times (c, s_1]$ preserving second coordinate and sending the graphs of functions $\gamma_{j_1}, \dots, \gamma_{j_2-1}$ onto vertical segments. Let us extend ϕ_1 by the identity on $\mathbb{R} \times [s_1, d]$ to a homeomorphism of all of $\mathbb{R} \times (c, d]$.

Denote by γ_i^1 the image of the graph of γ_i under ϕ_1 . Then again there exists a homeomorphism $\phi_2 : \mathbb{R} \times (c, d] \rightarrow \mathbb{R} \times (c, d]$ preserving second coordinate, fixed on $\mathbb{R} \times [s_2, d]$, and sending the graphs of functions $\gamma_{j_1}^1, \dots, \gamma_{j_3-1}^1$ onto vertical segments.

Hence the composition $\phi_2 \circ \phi_1$ preserves second coordinate and sends the graphs of functions $\gamma_{j_1}, \dots, \gamma_{j_3-1}$ onto vertical segments. Denote by γ_i^2 the graph of the function γ_i under $\phi_2 \circ \phi_1$.

Then by similar arguments, we will construct an infinite family of homeomorphisms $\phi_1, \dots, \phi_k, \dots$ of $\mathbb{R} \times (c, d]$ such that each ϕ_k preserves second coordinate, is fixed on $\mathbb{R} \times [s_k, d]$, and sends the graphs of functions $\gamma_{j_1}^1, \dots, \gamma_{j_{k+1}-1}^1$ onto vertical segments.

Since $\lim_{i \rightarrow \infty} s_i = c$, it follows that the infinite composition

$$\eta = \dots \circ \phi_m \circ \phi_{m-1} \circ \dots \circ \phi_1 : \mathbb{R} \times (c, d] \rightarrow \mathbb{R} \times (c, d]$$

is a well defined homeomorphism satisfying the statement of lemma. \square

To deduce Proposition 5.2 assume that T_i is defined by functions $\alpha_i, \beta_i : (c, d_i] \rightarrow \mathbb{R}$. Denote $\gamma_{2i-1} = \alpha_i$ and $\gamma_{2i} = \beta_i$. Then existence of η is guaranteed by Lemma 5.2.3. \square

Level-preserving homeomorphisms between trapezoids. Let $q_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$, $q_2(x, y) = y$, be the standard projection onto the second coordinate and

$$\begin{aligned} S &= \{(x, y) \in \mathbb{R} \mid \alpha(y) \leq x \leq \beta(y), \ a < y \leq b\}, \\ T &= \{(x, y) \in \mathbb{R} \mid \gamma(y) \leq x \leq \delta(y), \ c < y \leq d\} \end{aligned}$$

be two trapezoids with finite bases, where $\alpha, \beta : (a, b] \rightarrow \mathbb{R}$ and $\gamma, \delta : (c, d] \rightarrow \mathbb{R}$ are continuous functions such that $\alpha < \beta$ and $\gamma < \delta$.

Let $A \subset S$ and $B \subset T$ be two subsets. Then a map $\xi : A \rightarrow B$ will be called *level-preserving* whenever

$$q_2 \circ \xi(x, y) = q_2 \circ \xi(x', y)$$

for all x, x', y such that $(x, y), (x', y) \in A$.

Lemma 5.3. *Every level-preserving homeomorphism $\xi : \text{roof}(S) \rightarrow \text{roof}(T)$ between roofs of trapezoids extends to a level-preserving homeomorphism $\xi : S \rightarrow T$. Moreover, if S and T have finite bases, then ξ also extends to a level-preserving homeomorphism $\xi : \overline{S} \rightarrow \overline{T}$ between their closures.*

Proof. As ξ is level-preserving, we have a well defined homeomorphism $\sigma : (a, b] \rightarrow (c, d]$ given by $\sigma(y) = q_2 \circ \xi(\alpha(y), y)$. Then ξ extends to a homeomorphism $S \rightarrow T$ by

$$\xi(x, y) = \left(\gamma(\sigma(y)) + \frac{\delta(\sigma(y)) - \gamma(\sigma(y))}{\beta(y) - \alpha(y)}(x - \alpha(y)), \sigma(y) \right).$$

Moreover, if in addition S and T have finite bases, so α and β are defined and continuous on $[a, b]$ and γ and δ are defined and continuous on $[c, d]$, then the same formulas define homeomorphisms $\sigma : [a, b] \rightarrow [c, d]$ and $\xi : \bar{S} \rightarrow \bar{T}$. \square

6. PROOF OF (2) OF THEOREM 1.8

Let Δ be a partition on X of class \mathcal{F} such that the family Σ of all special leaves is locally finite. Let also $\bar{\Sigma} = \Sigma \cup \partial X$ be the union of all special and boundary leaves of X , Q be a connected component of $X \setminus \bar{\Sigma}$ homeomorphic with an open model strip, and $\phi : \mathbb{R} \times (-1, 1) \rightarrow Q$ be a foliated homeomorphism. Denote, see Figure 6.1:

$$(6.1) \quad A = \phi(\mathbb{R} \times (-1, 0]), \quad K = \phi(\mathbb{R} \times 0), \quad B = \phi(\mathbb{R} \times [0, 1)).$$

We should prove that the closures \bar{A} and \bar{B} are foliated homeomorphic to some model strips. It suffices to prove this only for \bar{A} .

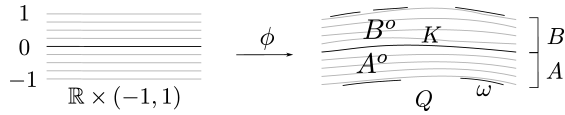


FIGURE 6.1.

Lemma 6.1. 1) $\bar{A} \setminus Q = \bar{A} \setminus A$ and $\bar{B} \setminus Q = \bar{B} \setminus B$.

2) $\bar{Q} \setminus Q = (\bar{A} \setminus A) \cup (\bar{B} \setminus B) \subset \bar{\Sigma}$.

3) Let ω be a leaf in $\bar{A} \setminus A$, $J = (-1, 1) \times 0 \subset \mathbb{R}^2$, and $N = J \cup \mathbb{R} \times (0, 1]$.

(a) Then $Q \cup \omega$ is open in \bar{Q} and $A \cup \omega$ is open in \bar{A} .

(b) There exists a foliated homeomorphism $\psi : N \rightarrow A \cup \omega$ such that $\psi(J) = \omega$.

(c) Let $U \subset X$ be an open neighborhood of ω and $T \subset \psi^{-1}(U)$ be a subset with compact closure such that

$$\bar{U} \cap (\bar{A} \setminus A) = \omega, \quad J \subset (\bar{T} \setminus T) \subset \mathbb{R} \times 0.$$

Then $\psi(T)$ is closed in X . In particular, if T is a trapezoid with base J , then $\psi(T \cup J)$ is closed in X .

Proof. 1) Denote $A^o = A \setminus K$ and $B^o = B \setminus K$. Then $K \subset \bar{A}^o$, so

$$\bar{A} = \overline{A^o \cup K} = \bar{A}^o \cup K = \bar{A}^o.$$

Moreover, as A^o and B^o are open in X and disjoint, we get that $\bar{A} \cap B^o = \bar{A}^o \cap B^o = \emptyset$, whence $\bar{A} \setminus Q = \bar{A} \setminus (A \cup B^o) = \bar{A} \setminus A$. The proof for B is similar.

2) It follows from (1) that $\bar{Q} \setminus Q = (\bar{A} \setminus Q) \cup (\bar{B} \setminus Q) = (\bar{A} \setminus A) \cup (\bar{B} \setminus B)$.

Let us prove that $\bar{Q} \setminus Q \subset \bar{\Sigma}$. Suppose $\bar{Q} \setminus Q \not\subset \bar{\Sigma}$. Then there exists a connected component P of $X \setminus \bar{\Sigma}$ distinct from Q and such that $\bar{Q} \cap P \neq \emptyset$. But P is open in X , whence $P \cap Q \neq \emptyset$ and so $P = Q$ which contradicts to the assumption.

3a) Notice that the family $\bar{\Sigma} \setminus \{\omega\}$ is locally finite as well as $\bar{\Sigma}$. Therefore the set

$$W := X \setminus (\bar{\Sigma} \setminus \omega) = (X \setminus \bar{\Sigma}) \cup \omega$$

is open in X . Due to 2), $Q = \bar{Q} \cap (X \setminus \bar{\Sigma})$, whence $Q \cup \omega = \bar{Q} \cap ((X \setminus \bar{\Sigma}) \cup \omega) = \bar{Q} \cap W$ is open in \bar{Q} .

Similarly, due to 1), $A = \overline{A} \cap Q$, whence $A \cup \omega = \overline{A} \cap \overline{Q} \cap ((X \setminus \overline{\Sigma}) \cup \omega) = \overline{A} \cap W$ is open in \overline{A} .

3b) Notice that $A \cup \omega$ is saturated and by Lemma 2.4 $p(A \cup \omega)$ is homeomorphic with $[0, 1]$. Since $p : A \cup \omega \rightarrow p(A \cup \omega)$ is a locally trivial fibration with fiber \mathbb{R} , we obtain that $A \cup \omega$ is foliated homeomorphic with $\mathbb{R} \times [0, 1]$ and therefore with N .

3c) It suffices to prove that $\psi(T)$ is closed in $\overline{A} \setminus U$ being a closed subset of X , which will imply that $\psi(T)$ is closed in X as well.

Let $\{z_i\}_{i \in \mathbb{N}} \subset \psi(T)$ be a sequence converging to some $z \in \overline{A}$. We should prove that $z \in \psi(T)$ as well. Let $(x_i, y_i) = \psi^{-1}(z_i) \in T$. Since \overline{T} is compact, one can assume that $\{(x_i, y_i)\}$ converges to some $(\bar{x}, \bar{y}) \in \overline{T}$.

If $(\bar{x}, \bar{y}) \in T$, then $z = \lim_{i \rightarrow \infty} z_i = \lim_{i \rightarrow \infty} \psi(x_i, y_i) = \psi(\bar{x}, \bar{y}) \in \psi(T)$. Otherwise, we have that $(\bar{x}, \bar{y}) \in \overline{T} \setminus T \subset \mathbb{R} \times 0$, so $\bar{y} = 0$, and thus $\lim_{i \rightarrow \infty} y_i = \bar{y} = 0$. This implies that $z \notin A = \psi(\mathbb{R} \times (0, 1])$. Hence $z \in \overline{U} \cap (\overline{A} \setminus A) = \omega = \psi(J) \subset \psi(T)$. \square

Due to (5) of Lemma 3.1 there exists a family $\mathcal{U} = \{U_\omega\}_{\omega \in \overline{\Sigma}}$ of neighborhoods of elements of $\overline{\Sigma}$ such that the closures of elements of \mathcal{U} are pairwise disjoint in X .

Let $\{\omega_i\}_{i \in \Lambda}$ be all the leaves contained in $\overline{A} \setminus A$. Then Λ is at most countable set and one can assume that either $\Lambda = \{1, \dots, k\}$ for some finite k or $\Lambda = \mathbb{N}$.

By Lemma 6.1 for each $i \in \Lambda$ there exists a foliated homeomorphism $\phi_i : N \rightarrow A \cup \omega_i$ such that $\psi_i(J) = \omega_i$.

Then $\psi_i^{-1}(U_{\omega_i})$ is an open neighborhood of $J = (-1, 1) \times 0$, whence by Lemma 5.1 there exists a trapezoid $T_i \subset \psi_i^{-1}(U_{\omega_i}) \cap \mathbb{R} \times (0, 1)$ with base J . Put

$$\widehat{T}_i = T_i \cup J.$$

Then by Lemma 6.1 $\psi_i(\widehat{T}_i)$ is closed in \overline{A} .

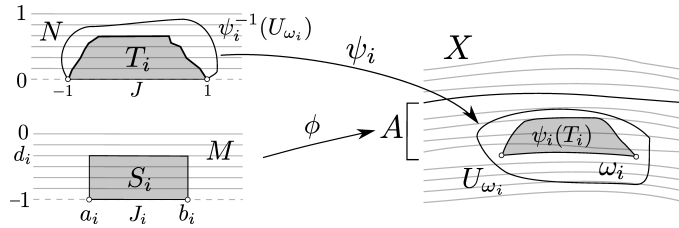


FIGURE 6.2.

Denote $S_i = \phi^{-1} \circ \psi_i(T_i)$. Then $\{S_i \mid i \in \Lambda\}$ is a family of trapezoids in $\mathbb{R} \times (-1, 0]$. Assume that the upper base of S_i is contained in $\mathbb{R} \times d_i$ for some $d_i \in (-1, 0)$. If Λ is infinite, then decreasing, if necessary, T_i (and therefore S_i), one can assume that $\lim_{i \rightarrow \infty} d_i = -1$. Then by Proposition 5.2 one can change ϕ so that $S_i = [a_i, b_i] \times (-1, d_i]$ is a “half open rectangle” for some $a_i, b_i \in \mathbb{R}$. Then $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ for $i \neq j \in \Lambda$. Let also $J_i = (a_i, b_i) \times \{-1\}$ and

$$M := \mathbb{R} \times (-1, 0] \bigcup_{i \in \Lambda} J_i.$$

Then M is a *half model strip*. Our aim is to construct a foliated homeomorphism between M and \overline{A} .

Denote $\widehat{S}_i = S_i \cup J_i$, $i \in \Lambda$, and

$$Z := M \setminus \bigcup_{i \in \Lambda} (\widehat{S}_i \setminus \text{roof}(S_i)) \subset \mathbb{R} \times (0, 1].$$

Lemma 6.2. $\{Z\} \cup \{\widehat{S}_i\}_{i \in \Lambda}$ is a locally finite cover of M by closed sets.

Proof. It is evident, that \widehat{S}_i is closed in M . Moreover $\widehat{S}_i \setminus \text{roof}(S_i)$ is open in M , whence Z is closed in M as well. Therefore it remains only to show that each $z = (x, y) \in M$ has an open neighborhood V intersecting only finitely many elements \widehat{S}_i .

If $y = -1$, then $z \in (a_i, b_i) \times \{-1\} \subset \widehat{S}_i$ for some $i \in \Lambda$. Hence $V = \widehat{S}_i \setminus \text{roof}(S_i)$ is an open neighborhood of z in N intersecting only \widehat{S}_i .

Suppose that $y > -1$. Fix any t such that $-1 < t < y$. Then $V = \mathbb{R} \times (t, 0]$ is an open neighborhood of z in M . By assumption $\lim_{i \rightarrow \infty} d_i = -1$, whence there exists $n > 0$ such that $-1 < d_i < t$ for all $i > n$, and so $\widehat{S}_i \cap V = \emptyset$. \square

Lemma 6.3. $\{\phi(Z)\} \cup \{\psi_i(\widehat{T}_i)\}_{i \in \Lambda}$ is a locally finite cover of \overline{A} by closed sets.

Proof. By 3c) of Lemma 6.1 each $\psi_i(\widehat{T}_i)$ is closed in X . Furthermore,

$$\phi(Z) = \phi\left(M \setminus \bigcup_{i \in \Lambda} (\widehat{S}_i \setminus \text{roof}(S_i))\right) = \overline{A} \setminus \bigcup_{i \in \Lambda} \psi_i(\widehat{T}_i \setminus \text{roof}(T_i)),$$

and it is also evident that $\widehat{T}_i \setminus \text{roof}(T_i)$ is open in N . But due to 3b) of Lemma 6.1 ψ_i is a homeomorphism of N onto the open subset $A \cup \omega_i$ of \overline{A} . Therefore $\psi_i(\widehat{T}_i \setminus \text{roof}(T_i))$ is open in \overline{A} , whence $\phi(Z)$ is closed in \overline{A} .

It remains to show that $\{\psi_i(\widehat{T}_i)\}_{i \in \Lambda}$ is a locally finite family. Let $q \in \overline{A}$.

If $q \in \omega_i$, then U_{ω_i} is an open neighborhood of q intersecting only one set $\psi_i(\widehat{T}_i)$.

Suppose $q \in \overline{A} \setminus A$ and let $z = (x, y) = \phi^{-1}(q) \in \mathbb{R} \times (-1, 0] \subset M$. Then by Lemma 6.2 there exists an open neighborhood V of z in $\mathbb{R} \times (-1, 0]$ intersecting only finitely many \widehat{S}_i . But the map $\phi : \mathbb{R} \times (-1, 0] \rightarrow A$ is a homeomorphism, whence $\phi(V)$ is an open neighborhood of q in A intersecting only finitely many $\psi_i(\widehat{T}_i) = \phi(S_i) \cup \omega_i$. \square

Notice that the composition $\psi^{-1} \circ \phi|_{S_i} : S_i \rightarrow T_i$ is a level-preserving homeomorphism, however in general it can not be extended to a homeomorphism between their bases. Nevertheless, $\psi^{-1} \circ \phi$ yields a level-preserving homeomorphism $\text{roof}(S_i) \rightarrow \text{roof}(T_i)$, and therefore by Lemma 5.3 it extends to a level-preserving homeomorphism $\xi_i : \overline{S}_i \rightarrow \overline{T}_i$.

Now define the following map $\eta : M \rightarrow \overline{A}$ by

$$\eta(z) = \begin{cases} \psi_i \circ \xi_i(z), & z \in \widehat{S}_i \text{ for some } i \in \Lambda, \\ \phi(z), & z \in Z. \end{cases}$$

We claim that η is the required homeomorphism.

Indeed, evidently, η is a bijection. Furthermore, by Lemma 6.2 $\{Z\} \cup \{\widehat{S}_i\}_{i \in \Lambda}$ is a locally finite closed cover of M , and by Lemma 6.3 their images $\{\phi(Z)\} \cup \{\psi_i(\widehat{T}_i)\}_{i \in \Lambda}$ constitute a locally finite closed cover of \overline{A} . Finally, the restrictions $\eta|_Z$ and $\eta|_{\widehat{S}_i}$ are homeomorphisms. Then by (6) of Lemma 3.1 η is a homeomorphism. Theorem 1.8 is completed.

REFERENCES

1. A. Andronov and L. Pontryagin, *Systèmes grossiers*, Dokl. Akad. Nauk. SSSR **14** (1937), 247–251. (Russian)
2. S. H. Aranson and V. Z. Grines, *On some invariants of dynamical systems on two-dimensional manifolds (necessary and sufficient conditions for the topological equivalence of transitive dynamical systems)*, Math. USSR Sb. **19** (1973), no. 3, 365–394.
3. S. Kh. Aranson and V. Z. Grines, *Topological classification of flows on closed two-dimensional manifolds*, Russ. Math. Surv. **41** (1986), no. 1, 183–208.
4. S. Kh. Aranson, V. Z. Grines, and V. A. Kaimanovich, *Classification of supertransitive 2-webs on surfaces*, J. Dynam. Control Systems **9** (2003), no. 4, 455–468.
5. S. Kh. Aranson, E. V. Zhuzhoma, and V. S. Medvedev, *On continuity of geodesic frameworks of flows on surfaces*, Sb. Math. **188** (1997), no. 7, 955–972.

6. William M. Boothby, *The topology of regular curve families with multiple saddle points*, Amer. J. Math. **73** (1951), 405–438.
7. William M. Boothby, *The topology of the level curves of harmonic functions with critical points*, Amer. J. Math. **73** (1951), 512–538.
8. Idel Bronstein and Igor Nikolaev, *Peixoto graphs of Morse-Smale foliations on surfaces*, Topology Appl. **77** (1997), no. 1, 19–36.
9. N. V. Budnyts'ka and O. O. Pryshlyak, *Equivalence of closed 1-forms on surfaces with edge*, Ukrainian Math. J. **61** (2009), no. 11, 1710–1727.
10. N. V. Budnyts'ka and T. V. Rybalkina, *Realization of a closed 1-form on closed oriented surfaces*, Ukrainian Math. J. **64** (2012), no. 6, 844–856.
11. Michael Farber, *Topology of closed one-forms*, Mathematical Surveys and Monographs, vol. 108, American Mathematical Society, Providence, RI, 2004.
12. James A. Jenkins and Marston Morse, *Contour equivalent pseudoharmonic functions and pseudoconjugates*, Amer. J. Math. **74** (1952), 23–51.
13. Wilfred Kaplan, *Regular curve-families filling the plane, I*, Duke Math. J. **7** (1940), 154–185.
14. Sergey Maksymenko, *Stabilizers and orbits of smooth functions*, Bull. Sci. Math. **130** (2006), no. 4, 279–311.
15. Sergiy Maksymenko and Eugene Polulyakh, *Foliations with non-compact leaves on surfaces*, Proc. Intern. Geom. Center **8** (2015), no. 3–4, 17–30.
16. M. Morse, *The existence of pseudoconjugates on Riemann surfaces*, Fund. Math. **39** (1952), 269–287 (1953).
17. A. A. Oshemkov and V. V. Sharko, *On the classification of Morse-Smale flows on two-dimensional manifolds*, Sb. Math. **189** (1998), no. 8, 1205–1250.
18. M. M. Peixoto, *Structural stability on two-dimensional manifolds*, Topology **1** (1962), 101–120.
19. M. M. Peixoto, *Structural stability on two-dimensional manifolds. A further remark*, Topology **2** (1963), 179–180.
20. L. P. Plachta, *The combinatorics of gradient-like flows and foliations on closed surfaces. II. The problem of realization and some estimates*, Mat. Metodi Fiz.-Mekh. Polya **44** (2001), no. 2, 7–16.
21. L. P. Plachta, *The combinatorics of gradient-like flows and foliations on closed surfaces. III. The problem of realization and some estimates*, Mat. Metodi Fiz.-Mekh. Polya **44** (2001), no. 3, 7–16.
22. Leonid Plachta, *The combinatorics of gradient-like flows and foliations on closed surfaces. I. Topological classification*, Topology Appl. **128** (2003), no. 1, 63–91.

INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES OF UKRAINE, 3 TERESHCHENKIVS'KA, KYIV, 01601, UKRAINE

E-mail address: Sergiy Maksymenko <maks@imath.kiev.ua>

INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES OF UKRAINE, 3 TERESHCHENKIVS'KA, KYIV, 01601, UKRAINE

E-mail address: Eugene Polulyakh <polulyah@imath.kiev.ua>

Received 30/05/2016